Eulerian polynomials, chromatic quasisymmetric functions, and Hessenberg varieties

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Joint work with John Shareshian
Eulerian Polynomials

For $\sigma \in S_n$

$$\text{DES}(\sigma) := \{i \in \{1, \ldots, n-1\} : \sigma(i) > \sigma(i+1)\}$$

$$\text{des}(\sigma) := |\text{DES}(\sigma)|$$

For $\sigma = 3.25.4.1$

$$\text{DES}(\sigma) = \{1, 3, 4\} \quad \text{des}(\sigma) = 3$$

Eulerian polynomial

$$A_n(t) = \sum_{\sigma \in S_n} t^{\text{des}(\sigma)}$$

$$A_1(t) = 1$$
$$A_2(t) = 1 + t$$
$$A_3(t) = 1 + 4t + t^2$$
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- palindromic
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A new formula?

\[ A_n(t) = \sum_{m=1}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \sum_{k_1, \ldots, k_m \geq 2} \left( \begin{array}{c} n \\ k_1 - 1, k_2, \ldots, k_m \end{array} \right) t^{m-1} \prod_{i=1}^{m} [k_i - 1]_t \]

where

\[ [k]_t := 1 + t + \cdots + t^{k-1}. \]
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Sum & Product Lemma: Let \( A(t) \) and \( B(t) \) be positive, unimodal, palindromic with respective centers of symmetry \( c_A \) and \( c_B \). Then

- \( A(t)B(t) \) is positive, unimodal, and palindromic with center of symmetry \( c_A + c_B \).
- If \( c_A = c_B \) then \( A(t) + B(t) \) is positive, unimodal and palindromic with center of symmetry \( c_A \).
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\[ A_n(t) = \sum_{m=1}^{\lfloor n+1/2 \rfloor} \sum_{k_1, \ldots, k_m \geq 2} \binom{n}{k_1 - 1, k_2, \ldots, k_m} t^{m-1} \prod_{i=1}^{m} [k_i - 1]_t \]

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**Center of symmetry:**

\[(m-1) + \sum_{i=1}^{m} \frac{k_i - 2}{2} = \frac{1}{2} (n-1).\]
\[1 + \sum_{n \geq 1} A_n(t) \frac{z^n}{n!} = \frac{1 - t}{e^{z(t-1)} - t}\]

(Euler’s exponential generating function formula)

\[= \frac{(1 - t) \exp(z)}{\exp(zt) - t \exp(z)}\]

\[= \exp(z) \frac{1 - t}{\sum_{k \geq 0} \frac{t^k z^k - tz^k}{k!}}\]

\[= \exp(z) \frac{1 - t}{\sum_{k \geq 0} \frac{(t^k - t)z^k}{k!}}\]

\[= \left( \sum_{r \geq 0} \frac{z^r}{r!} \right) \left( 1 - \sum_{k \geq 2} \frac{t[k - 1]z^k}{k!} \right)^{-1}\]

\[= \left( \sum_{r \geq 0} \frac{z^r}{r!} \right) \sum_{m \geq 0} \left( \sum_{k \geq 2} \frac{t[k - 1]z^k}{k!} \right)^m\]

\[A_n(t) = \sum_{m=0}^{\lfloor n/2 \rfloor} \sum_{r=0}^n \sum_{k_1, \ldots, k_m \geq 2} \binom{n}{r, k_1, \ldots, k_m} t^m \prod_{i=1}^m [k_i - 1]_t\]

= further manipulations

\[= \sum_{m=1}^{\lfloor (n+1)/2 \rfloor} \sum_{k_1, \ldots, k_m \geq 2} \binom{n}{k_1 - 1, k_2, \ldots, k_m} t^{m-1} \prod_{i=1}^m [k_i - 1]_t\]
Mahonian Permutation Statistics - q-analog

Let $\sigma \in \mathfrak{S}_n$.

**Inversion Number:**

$$\text{inv}(\sigma) := |\{(i, j) : 1 \leq i < j \leq n, \quad \sigma(i) > \sigma(j)\}|.$$  

$$\text{inv}(32541) = 6$$

**Major Index:**

$$\text{maj}(\sigma) := \sum_{i \in \text{DES}(\sigma)} i$$

$$\text{maj}(32541) = \text{maj}(3.25.4.1) = 1 + 3 + 4 = 8$$
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**Theorem (MacMahon 1905)**
\[
\sum_{\sigma \in \mathfrak{S}_n} q^{\text{inv}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} q^{\text{maj}(\sigma)} = [n]_q!
\]

where \([n]_q := 1 + q + \cdots + q^{n-1}\) and \([n]_q! := [n]_q[n-1]_q \cdots [1]_q\).
Let $1 \leq r \leq n$. For $\sigma \in \mathfrak{S}_n$, set

$$\text{inv}_{<r}(\sigma) := |\{(i,j) : 1 \leq i < j \leq n, \quad 0 < \sigma(i) - \sigma(j) < r\}|.$$  

$$\text{DES}_{\geq r}(\sigma) := \{i \in [n-1] : \sigma(i) - \sigma(i+1) \geq r\}$$  

$$\text{maj}_{\geq r}(\sigma) := \sum_{i \in \text{DES}_{\geq r}} i$$

Note

$$\text{maj}_{\geq r} + \text{inv}_{<r} = \begin{cases} 
\text{maj} & \text{if } r = 1 \\
\text{inv} & \text{if } r = n
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**Theorem (Rawlings, 1981)**

$$\sum_{\sigma \in S_n} q^{\text{maj}_{\geq r}(\sigma) + \text{inv}_{<r}(\sigma)} = [n]_q!$$
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A^{(r)}_n(q, t) := \sum_{\sigma \in S_n} q^{\text{maj}_{\geq r}(\sigma)} t^{\text{inv}_{< r}(\sigma)}
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\[ A_n^{(r)}(q, t) := \sum_{\sigma \in \mathfrak{S}_n} q^{\text{maj} \geq r(\sigma)} t^{\text{inv} < r(\sigma)} \]

- \( A_n^{(1)}(q, t) = \sum_{\sigma \in \mathfrak{S}_n} q^{\text{maj}(\sigma)} = [n]_q! \)
- \( A_n^{(n)}(q, t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{inv}(\sigma)} = [n]_t! \)
- \( A_n^{(2)}(q, t) =? \)
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$\text{inv}_2(635142) = 3$ since $(< 2)$-inversions are

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So \( A_n^{(r)}(1, t) \) is a generalized Eulerian polynomial and \( A_n^{(r)}(q, qt) \) is a Mahonian \( q \)-analog.
Generalized Eulerian polynomial

\[ A_n^{(r)}(t) := A_n^{(r)}(1, t) = \sum_{\sigma \in \mathcal{S}_n} t^{\text{inv}_< r(\sigma)} \]

Eulerian polynomials are \textit{palindromic and unimodal}.

\[ A_3^{(2)}(t) = 1 + 4t + t^2 \]

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\[ A^{(n-1)}_n(t) = [n-2]_t!([n]_t[n-2]_t + nt^{n-2}) \]

center of symmetry: \( \frac{1}{2}(((n-1) + (n-3)) = n - 2 \)
Generalized Eulerian polynomial $A_n^{(r)}(t) = \sum_{\sigma \in \mathfrak{S}_n} t^{\text{inv}_{<r}(\sigma)}$

Problem (Stanley EC1, 1.50 f): Prove that $A_n^{(r)}(t)$ is palindromic and unimodal.

Solution:
Generalized Eulerian polynomial $A_n^{(r)}(t) = \sum_{\sigma \in S_n} t^{{\text{inv}}_<(\sigma)}$

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Theorem (De Mari and Shayman (1988))

$A_n^{(r)}(t)$ is palindromic and unimodal for all $r \in [n]$.

- Palindromicity: easy
- Unimodality: They show that $A_n^{(r)}(t)$ is the Poincaré polynomial of a Hessenberg variety and apply hard Lefschetz theorem.

Stanley: Is there a more elementary proof?
Generalized Eulerian polynomial \( A_n^{(r)}(t) = \sum_{\sigma \in S_n} t^{\text{inv} <_r (\sigma)} \)

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Is there a \( q \)-analog of this result?
q-analog: \( A_n^{(r)}(q, t) = \sum_{\sigma \in \mathfrak{S}_n} q^{maj \geq r(\sigma)} t^{inv < r(\sigma)} \)

A polynomial \( \sum_{j=0}^n f_j(q) t^j \in \mathbb{Z}[q][t] \) is q-unimodal if

\[
f_0(q) \leq_q f_1(q) \leq_q \cdots \leq_q f_c(q) \geq_q \cdots \geq_q f_{n-1}(q) \geq_q f_n(q),
\]

where \( f(q) \leq_q g(q) \) means \( g(q) - f(q) \in \mathbb{N}[q] \).
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**Proposition**

$A_n^{(r)}(q, t)$ is palindromic as a polynomial in $t$ for all $r \in [n]$.

**Conjecture (Shareshian and MW)**

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A_n^{(n-1)}(q, t) = [n-2]_t ([n]_t [n-2]_t + [n]_q t^{n-2})
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$q$-Eulerian polynomials, $r = 2$:

$$A_n^{(2)}(q, t) = \sum_{\sigma \in \mathfrak{S}_n} q^{\text{maj}_{\geq 2}(\sigma)} t^{\text{inv}_{< 2}(\sigma)}$$

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**q-Eulerian polynomials, r = 2:**

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**Theorem (Shareshian and MW)**

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A_n^{(2)}(q, t) = \sum_{m=1}^{n+1} \sum_{k_1, \ldots, k_m \geq 2} \left[ \begin{array}{c}
\frac{n+1}{2} \\
k_1 - 1, k_2, \ldots, k_m
\end{array} \right]_q t^{m-1} \prod_{i=1}^{m} [k_i - 1]_t
\]

To prove this we use theory of P-partitions and quasisymmetric functions to get a q-analog of Euler's exponential generating function formula.
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<td>3</td>
<td>1</td>
<td>2 + q + q^2</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>3 + 2q + 3q^2 + 2q^3 + q^4</td>
<td>3 + 2q + 3q^2 + 2q^3 + q^4</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>4 + 3q + 5q^2 + ...</td>
<td>6 + 6q + 11q^2 + ...</td>
<td>4 + 3q + 5q^2 + ...</td>
<td>1</td>
</tr>
</tbody>
</table>

**Theorem (Shareshian and MW)**

\[
A_{n}^{(2)}(q, t) = \sum_{m=1}^{\left\lfloor \frac{n+1}{2} \right\rfloor} \sum_{k_1, \ldots, k_m \geq 2} \left[ \begin{array}{c} n \\ k_1 - 1, k_2, \ldots, k_m \end{array} \right] q^{t^{m-1} \prod_{i=1}^{m} [k_i - 1]} t^m
\]

Shareshian & MW: \( A_{n}^{(2)}(q, t) = \sum_{\sigma \in S_n} q^{\text{maj}(\sigma) - \text{exc}(\sigma)} t^{\text{exc}(\sigma)} \)
### Formulae

<table>
<thead>
<tr>
<th>$r$</th>
<th>$A_n^{(r)}(q, t)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$[n]_q!$</td>
</tr>
</tbody>
</table>
| 2   | \[
\sum_{m=1}^{\left\lceil \frac{n+1}{2} \right\rceil} \sum_{k_1, \ldots, k_m \geq 2} \left[ k_1 - 1, k_2, \ldots, k_m \right]_q t^{m-1} \prod_{i=1}^{m} [k_i - 1]_t
\] |
| $n-2$ | $[n]_t [n-3]_t! [n-3]^2_t + [n]_q t^{n-3} [n-4]_t! [n-2]_t ([n-3]_t + [2]_t [n-4]_t)$ + $\left[ \begin{array}{c} n \\ n-2, 2 \end{array} \right]_q t^{3n-10} [n-4]! [n-2]_t [2]_t$ |
| $n-1$ | $[n]_t [n-2]_t! [n-2]_t + [n]_q t^{n-2} [n-2]_t!$ |
| $n$  | $[n]_t!$         |
The $q$-unimodality conjecture - again

Conjecture (Shareshian and MW)

$A_n^{(r)}(q, t)$ is $q$-unimodal for all $r \in [n]$.

True for $q = 1$ Hessenberg varieties

True for $1 \leq r \leq 2$ & $n - 2 \leq r \leq n$ Sum & Product Lemma
Let $\mathcal{F}$ be the set of all flags

$$F : V_1 \subset V_2 \subset \cdots \subset V_n = \mathbb{C}^n$$

where $\dim V_i = i$. Fix $M \in GL_n(\mathbb{C})$ with $n$ distinct eigenvalues.

The type A regular semisimple Hessenberg variety of degree $r$ is

$$\mathcal{H}_{n,r} := \{F \in \mathcal{F} \mid MV_i \subseteq V_{i+r-1} \text{ for all } i\}$$

**Theorem (De Mari and Shayman - 1988)**

$$A_n^{(r)}(1, t) = \sum_{j=0}^{d(n,r)} \dim H^{2j}(\mathcal{H}_{n,r}) t^j$$

Consequently by the hard Lefschetz theorem, $A_n^{(r)}(1, t)$ is palindromic and unimodal.
The hard Lefschetz theorem

Theorem (Hard Lefschetz Theorem)

Let $Y$ be a smooth irreducible complex projective variety of (complex) dimension $m$. Then for some $\omega \in H^2(Y)$ and all $j = 0, \ldots, m$, the map $H^j(Y) \to H^{2m-j}(Y)$, given by multiplication by $\omega^{m-j}$ in the singular cohomology ring $H^*(Y)$, is a vector space isomorphism.

It follows that the map $H^j(Y) \to H^{j+2}(Y)$ given by multiplication by $\omega$ is injective. Hence for all $j = 0, \ldots, m$,

$$\dim H^j(Y) \leq \dim H^{j+2}(Y)$$

$$H^j(Y) = H^{2m-j}(Y)$$
The hard Lefschetz theorem

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$$\dim H^j(Y) \leq \dim H^{j+2}(Y)$$

$$H^j(Y) = H^{2m-j}(Y)$$

Consequently the Poincaré polynomial $\sum_{j=0}^{m} \dim H^{2j}(Y) t^j$ is palindromic and unimodal.
The hard Lefschetz theorem

**Theorem (Hard Lefschetz Theorem)**

Let $Y$ be a smooth irreducible complex projective variety of (complex) dimension $m$. Then for some $\omega \in H^2(Y)$ and all $j = 0, \ldots, m$, the map $H^j(Y) \to H^{2m-j}(Y)$, given by multiplication by $\omega^{m-j}$ in the singular cohomology ring $H^*(Y)$, is a vector space isomorphism.

It follows that the map $H^j(Y) \to H^{j+2}(Y)$ given by multiplication by $\omega$ is injective. Hence for all $j = 0, \ldots, m$,

$$\dim H^j(Y) \leq \dim H^{j+2}(Y)$$

$$H^j(Y) = H^{2m-j}(Y)$$

Consequently the Poincaré polynomial $\sum_{j=0}^m \dim H^{2j}(Y) t^j$ is palindromic and unimodal. Recall

$$A_n^{(r)}(1, t) = \sum_{j=0}^{d(n,r)} \dim H^{2j}(\mathcal{H}_{n,r}) t^j$$
From representations to $q$-analogs

\[
\{\text{Representations of } \mathfrak{S}_n\} \xrightarrow{\text{ch}} \Lambda^n_\mathbb{Z} \xrightarrow{\text{ps}} \mathbb{Z}[q]
\]

$\Lambda^n_\mathbb{Z}$: homogeneous symmetric functions over $\mathbb{Z}$ of degree $n$

ch: Frobenius characteristic

ps: stable principal specialization.

\[
\text{ps}(f(x_1, x_2, \ldots)) = f(1, q, q^2, \ldots) \prod_{i=1}^{n}(1 - q^i)
\]
From representations to $q$-analogs

\{\text{Representations of } S_n\} \xrightarrow{\text{ch}} \Lambda^n_Z \xrightarrow{\text{ps}} \mathbb{Z}[q]

$\Lambda^n_Z$: homogeneous symmetric functions over $\mathbb{Z}$ of degree $n$

$\text{ch}$: Frobenius characteristic

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$$\text{ps}(f(x_1, x_2, \ldots)) = f(1, q, q^2, \ldots) \prod_{i=1}^{n} (1 - q^i)$$

**Proposition**

*If a symmetric function $f$ is Schur-positive then $\text{ps}(f)$ has nonnegative coefficients.*
From representations to $q$-analogs

\[
\{\text{Representations of } \mathfrak{S}_n\} \xrightarrow{\text{ch}} \Lambda^n_{\mathbb{Z}} \xrightarrow{\text{ps}} \mathbb{Z}[q]
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\text{ps}(f(x_1, x_2, \ldots)) = f(1, q, q^2, \ldots) \prod_{i=1}^{n}(1 - q^i)
\]

**Proposition**

*If a symmetric function $f$ is Schur-positive then $\text{ps}(f)$ has nonnegative coefficients.*

Tymoczko (2008) used a theory of Goresky, Kottwitz and MacPherson (GKM theory) to define a representation of $\mathfrak{S}_n$ on $H^2j(\mathcal{H}_n,r)$. 
MacPherson & Tymoczko show that the hard Lefschetz map commutes with the action of \( S_n \) on \( H^*(\mathcal{H}_{n,r}) \). This means that the hard Lefschetz map \( H^j(\mathcal{H}_{n,r}) \to H^{j+2}(\mathcal{H}_{n,r}) \) is an \( S_n \)-module injection for all \( j = 0, \ldots, d(n, r) \).

\[ \Rightarrow \text{ch} H^{j+2}(\mathcal{H}_{n,r}) - \text{ch} H^j(\mathcal{H}_{n,r}) \text{ is Schur-positive.} \]
MacPherson & Tymoczko show that the hard Lefschetz map commutes with the action of $\mathfrak{S}_n$ on $H^*(\mathcal{H}_{n,r})$. This means that the hard Lefschetz map $H^j(\mathcal{H}_{n,r}) \to H^{j+2}(\mathcal{H}_{n,r})$ is an $\mathfrak{S}_n$-module injection for all $j = 0, \ldots, d(n,r)$.

$\Rightarrow \ \text{ch} H^{j+2}(\mathcal{H}_{n,r}) - \text{ch} H^j(\mathcal{H}_{n,r}) \text{ is Schur-positive.}$

$\Rightarrow \ ps(\text{ch} H^{j+2}(\mathcal{H}_{n,r})) - ps(\text{ch} H^j(\mathcal{H}_{n,r})) \in \mathbb{N}[q]$. 

Conjecture (Shareshian and MW)

$A(r,n)(q,t) = d(n,r) \sum_{j=0}^{d(n,r)} ps(\text{ch} H^{2j}(\mathcal{H}_{n,r})) t^j$
MacPherson & Tymoczko show that the hard Lefschetz map commutes with the action of $\mathfrak{S}_n$ on $H^*(\mathcal{H}_{n,r})$. This means that the hard Lefschetz map $H^j(\mathcal{H}_{n,r}) \to H^{j+2}(\mathcal{H}_{n,r})$ is an $\mathfrak{S}_n$-module injection for all $j = 0, \ldots, d(n, r)$.

\[ \Rightarrow \text{ch}H^{j+2}(\mathcal{H}_{n,r}) - \text{ch}H^j(\mathcal{H}_{n,r}) \text{ is Schur-positive.} \]

\[ \Rightarrow ps(\text{ch}H^{j+2}(\mathcal{H}_{n,r})) - ps(\text{ch}H^j(\mathcal{H}_{n,r})) \in \mathbb{N}[q]. \]

Hence $\sum_{j=0}^{d(n,r)} ps(\text{ch}H^{2j}(\mathcal{H}_{n,r}))t^j$ is $q$-unimodal.
MacPherson & Tymoczko show that the hard Lefschetz map commutes with the action of $S_n$ on $H^*(\mathcal{H}_{n,r})$. This means that the hard Lefschetz map $H^j(\mathcal{H}_{n,r}) \to H^{j+2}(\mathcal{H}_{n,r})$ is an $S_n$-module injection for all $j = 0, \ldots, d(n,r)$.

$\Rightarrow$ $\text{ch} H^{j+2}(\mathcal{H}_{n,r}) - \text{ch} H^j(\mathcal{H}_{n,r})$ is Schur-positive.

$\Rightarrow$ $ps(\text{ch} H^{j+2}(\mathcal{H}_{n,r})) - ps(\text{ch} H^j(\mathcal{H}_{n,r})) \in \mathbb{N}[q]$.

Hence $\sum_{j=0}^{d(n,r)} ps(\text{ch} H^{2j}(\mathcal{H}_{n,r})) t^j$ is $q$-unimodal.

**Conjecture (Shareshian and MW)**

$$A_n^{(r)}(q, t) = \sum_{j=0}^{d(n,r)} ps(\text{ch} H^{2j}(\mathcal{H}_{n,r})) t^j$$
Let $G$ be a graph on $[n]$. The chromatic quasisymmetric function $X_G(x, t)$ is a polynomial in $t$ whose coefficients are quasisymmetric functions. It is a refinement of Stanley’s chromatic symmetric function $X_G(x)$. 

Theorem (Shareshian and MW)

$$A(n, q, t) = ps(\omega X_G n, r(x, t))$$

Conjecture (Shareshian and MW)

$$\omega X_G n, r(x, t) = d(n, r) \sum_{j=0}^{\infty} ch H_{2j}(H^n, r) t^j$$

True for $1 \leq r \leq 2$ and $n-2 \leq r \leq n$. ($r = 2$ follows from results of Procesi and Stanley)
Chromatic quasisymmetric functions

Let $G$ be a graph on $[n]$. The chromatic quasisymmetric function $X_G(x, t)$ is a polynomial in $t$ whose coefficients are quasisymmetric functions. It is a refinement of Stanley’s chromatic symmetric function $X_G(x)$.

For a certain class of graphs the coefficients are symmetric functions.

**Theorem (Shareshian and MW)**

$$A_n^{(r)}(q, t) = ps(\omega X_{G_n,r}(x, t))$$

**Conjecture (Shareshian and MW)**

$$\omega X_{G_n,r}(x, t) = \sum_{j=0}^{d(n,r)} \text{ch} H_{2j}(\mathcal{H}_{n,r}) t^j$$

True for $1 \leq r \leq 2$ \& $n - 2 \leq r \leq n$.

($r = 2$ follows from results of Procesi and Stanley.)
Let \( V(G) = \{1, 2, \ldots, n\} \). Let \( C(G) \) be the set of proper colorings of \( G \), where a proper coloring is a map \( c: V(G) \rightarrow P \) such that \( c(i) \neq c(j) \) if \( \{i, j\} \in E(G) \).

**Chromatic symmetric function** (Stanley, 1995)

\[
X_G(x) := \sum_{c \in C(G)} x^{c(1)} x^{c(2)} \ldots x^{c(n)}
\]

\[X_G(1, 1, \ldots, 1) \cdot 0 \cdot 0 \ldots = \chi_G(m) \]
Let $V(G) = \{1, 2, \ldots, n\}$. Let $C(G)$ be the set of proper colorings of $G$, where a proper coloring is a map $c: V(G) \rightarrow \mathbb{P}$ such that $c(i) \neq c(j)$ if $\{i, j\} \in E(G)$.

The chromatic symmetric function (Stanley, 1995) is defined as:

$$X_G(x) := \sum_{c \in C(G)} x^{c(1)} x^{c(2)} \cdots x^{c(n)}$$

Let $X_G(1, 1, \ldots, 1, 0, 0, \ldots) = \chi_G(m)$.
Let $V(G) = \{1, 2, \ldots, n\}$. Let $C(G)$ be the set of proper colorings of $G$, where a proper coloring is a map $c : V(G) \to \mathbb{P}$ such that $c(i) \neq c(j)$ if $\{i, j\} \in E(G)$. 
Let $V(G) = \{1, 2, \ldots, n\}$. Let $C(G)$ be set of proper colorings of $G$, where a proper coloring is a map $c : V(G) \to \mathbb{P}$ such that $c(i) \neq c(j)$ if $\{i, j\} \in E(G)$.

Chromatic symmetric function (Stanley, 1995)

$$X_G(x) := \sum_{c \in C(G)} x_{c(1)}x_{c(2)} \cdots x_{c(n)}$$
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**Chromatic symmetric function (Stanley, 1995)**

\[
X_G(x) := \sum_{c \in C(G)} x_{c(1)}x_{c(2)} \cdots x_{c(n)}
\]

\[
X_G(1, 1, \ldots, 1, 0, 0, \ldots) = \chi_G(m)
\]
Stanley-Stembridge Conjecture

A symmetric function $f(x)$ is said to be $e$-positive if its expansion in the basis of elementary symmetric functions $e_\lambda$ has nonnegative coefficients.

**Conjecture (Stanley, Stembridge 1993)**

*If $G$ is the incomparability graph of a $(3 + 1)$-free poset then $X_G$ is $e$-positive.*

\[ G = \begin{array}{ccc} 1 & 2 & 3 \end{array} \quad P = \begin{array}{c} 3 \\ 2 \\ 1 \end{array} \]

\[ X_G(x) = 3e_3 + e_{2,1} \]
Stanley-Stembridge Conjecture

A symmetric function $f(x)$ is said to be **e-positive** if its expansion in the basis of elementary symmetric functions $e_\lambda$ has nonnegative coefficients.

**Conjecture (Stanley, Stembridge 1993)**

*If $G$ is the incomparability graph of a $(3 + 1)$-free poset then $X_G$ is e-positive.*

\[ G = \begin{array}{c}
1 \\
2 \\
3 
\end{array} \quad P = \begin{array}{c}
1 \\
2
\end{array} \]

\[ X_G(x) = 3e_3 + e_{2,1} \]

**Theorem (Gasharov, 1996)**

$X_G$ is **Schur-positive**. The coefficient of $s_\lambda$ is the number of $P$-tableaux of shape $\lambda$. 
Chromatic quasisymmetric function

\[ X_G(x, t) := \sum_{c \in C(G)} t^{\text{des}(c)} x_{c(1)} x_{c(2)} \cdots x_{c(n)} \]

where

\[ \text{des}(c) := |\{ \{i, j\} \in E(G): i < j \text{ and } c(i) > c(j) \}|. \]
Chromatic quasisymmetric function

\[ G = \begin{array}{ccc}
1 & \rightarrow & 2 \\
\end{array} \begin{array}{cc}
\rightarrow & 3 \\
\end{array} \]

\[ X_G(x, t) = e_3 + (e_3 + e_{2,1})t + e_3 t^2 \]

\[ G = \begin{array}{ccc}
1 & \rightarrow & 3 \\
\end{array} \begin{array}{cc}
\rightarrow & 2 \\
\end{array} \]

\[ X_G(x, t) = \left( e_3 + F_{3,\{1\}} \right) + 2e_3 t + \left( e_3 + F_{3,\{2\}} \right) t^2 \]
Chromatic quasisymmetric function

\[ G = 1 \rightarrow 2 \rightarrow 3 \]
\[ P = 2 \rightarrow 1 \]

\[ X_G(x, t) = e_3 + (e_3 + e_{2,1})t + e_3 t^2 \]

\[ G = 1 \rightarrow 3 \rightarrow 2 \]
\[ P = 3 \rightarrow 1 \]

\[ X_G(x, t) = (e_3 + F_{3,\{1\}}) + 2e_3 t + (e_3 + F_{3,\{2\}}) t^2 \]
Let $\Lambda$ be the ring of symmetric functions over $\mathbb{Z}$. Unit interval orders are posets that are $(3+1)$-free and $(2+2)$-free.

**Theorem (Shareshian and MW)**

If $G$ is the incomparability graph of a natural unit interval order then

- $X_G(x, t) \in \Lambda[t]$
- $X_G(x, t)$ is palindromic as a polynomial in $t$. 

A polynomial $f(x, t) = \sum_{d \geq 0} a_d(x) t^d$ in $\Lambda[t]$ is $e$-positive if all the coefficients $a_d(x)$ are $e$-positive.

$e$-unimodal if $a_j(x) - a_{j-1}(x)$ is $e$-positive for all $1 \leq j \leq m$ and $a_j(x) - a_{j+1}(x)$ is $e$-positive for all $m \leq j \leq d$. 
Chromatic quasisymmetric function

Let \( \Lambda \) be the ring of symmetric functions over \( \mathbb{Z} \).

Unit interval orders are posets that are \((3 + 1)\)-free and \((2 + 2)\)-free.

**Theorem (Shareshian and MW)**

If \( G \) is the incomparability graph of a natural unit interval order then

- \( X_G(x, t) \in \Lambda[t] \)
- \( X_G(x, t) \) is palindromic as a polynomial in \( t \).

\[
X_{\{1,2,3\}}(x, t) = e_3 + (e_3 + e_{2,1})t + e_3t^2
\]
Let $\Lambda$ be the ring of symmetric functions over $\mathbb{Z}$. Unit interval orders are posets that are $(3+1)$-free and $(2+2)$-free.

**Theorem (Shareshian and MW)**

If $G$ is the incomparability graph of a natural unit interval order then

- $X_G(x, t) \in \Lambda[t]$ 
- $X_G(x, t)$ is palindromic as a polynomial in $t$.

\[
X_{\circlearrowleft 1 \circlearrowleft 2 \circlearrowleft 3}(x, t) = e_3 + (e_3 + e_{2,1})t + e_3t^2
\]

A polynomial $f(x, t) = \sum_{j=0}^{d} a_j(x)t^j$ in $\Lambda[t]$ is

- **e-positive** if all the coefficients $a_j(x)$ are e-positive.
- **e-unimodal** if $a_j(x) - a_{j-1}(x)$ is e-positive for all $1 \leq j \leq m$ and $a_j(x) - a_{j+1}(x)$ is e-positive for all $m \leq j \leq d$.
Conjecture (Shareshian and MW)

Let $G$ be the incomparability graph of a natural unit interval order. Then $X_G(x, t)$ is $e$-positive and $e$-unimodal.
Refinement of Stanley-Stembridge Conjecture

**Conjecture (Shareshian and MW)**

Let $G$ be the incomparability graph of a natural unit interval order. Then $X_G(x, t)$ is $e$-positive and $e$-unimodal.

Let $G_{n,r}$ be the graph with vertex set $\{1, 2, \ldots, n\}$ and edge set $\{\{i, j\} \mid 0 < |j - i| < r\}$.

<table>
<thead>
<tr>
<th>$r$</th>
<th>$X_{G_{n,r}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$e_1^n$</td>
</tr>
<tr>
<td>2</td>
<td>$\sum_{m=1}^{\lfloor \frac{n+1}{2} \rfloor} \sum_{k_1, \ldots, k_m \geq 2}^{\sum k_i = n+1} e_{k_1-1, k_2, \ldots, k_m} t^{m-1} \prod_{i=1}^{m} [k_i - 1]_t$</td>
</tr>
<tr>
<td>$\vdots$</td>
<td>$\ldots$</td>
</tr>
<tr>
<td>$n-2$</td>
<td>$e_n[n]_t[n-3]_t[n-3]<em>t^2 + e</em>{n-1,1} t^{n-3}[n-4]_t[n-2]_t([n-3]_t + [2]_t[n-4]<em>t) + e</em>{n-2,2} t^{3n-10}[n-4]_t[n-2]_t[2]_t$</td>
</tr>
<tr>
<td>$n-1$</td>
<td>$e_n[n]_t[n-2]_t[n-2]<em>t + e</em>{n-1,1} t^{n-2}[n-2]_t!$</td>
</tr>
<tr>
<td>$n$</td>
<td>$e_n[n]_t!$</td>
</tr>
</tbody>
</table>
Theorem (Shareshian and MW)

Let $G_{n,r}$ be the graph with vertex set $\{1, 2, \ldots, n\}$ and edge set $
\{\{i, j\} : 0 < j - i < r\}$. Then

$$A_n^{(r)}(q, t) = ps(\omega X_{G_{n,r}}(x, t))$$
Theorem (Shareshian and MW)

Let $G_{n,r}$ be the graph with vertex set $\{1, 2, \ldots, n\}$ and edge set $\{\{i, j\} : 0 < j - i < r\}$. Then

$$A_n^{(r)}(q, t) = ps(\omega X_{G_{n,r}}(x, t))$$

Theorem (Shareshian and MW - refinement of Gasharov)

Let $G$ be the incomparability graph of a natural unit interval order $P$. Then $X_G(x, t)$ is Schur-positive. Moreover for each $\lambda$ the coefficient of $s_\lambda$ is

$$\sum_{T \in \mathcal{T}_{P,\lambda}} t^{\text{inv}_G(T)}$$

where $\mathcal{T}_{P,\lambda}$ is the set of $P$-tableaux of shape $\lambda$.

Schur-unimodality is open.

Schur-unimodality $\Rightarrow$ unimodality conjecture for $A_n^{(r)}(q, t)$.
Conjecture (Shareshian and MW)

Let $G$ be the incomparability graph of a natural unit interval order. Then

$$
\omega X_G(x, t) = \sum_{j=0}^{d(n,r)} \text{ch}H^2j(\mathcal{H}_G)t^j
$$

$\mathcal{H}_{G_n,r} = \mathcal{H}_{n,r}$. True for $1 \leq r \leq 2$ & $n - 2 \leq r \leq n$.

($r = 2$ follows from results of Procesi and Stanley)
Conjecture (Shareshian and MW)

Let $G$ be the incomparability graph of a natural unit interval order. Then

$$
\omega X_G(x, t) = \sum_{j=0}^{d(n,r)} \text{ch} H^2j(H_G) t^j
$$

$H_{G_{n,r}} = H_{n,r}$. True for $1 \leq r \leq 2$ & $n - 2 \leq r \leq n$.

($r = 2$ follows from results of Procesi and Stanley)

$\Rightarrow$ Schur-unimodality conjecture for $X_G(x, t)$

$\Rightarrow$ $q$-unimodality conjecture for $A_G(q, t)$ (includes $A^{(r)}_{n}(q, t)$)

$\Rightarrow$ multiplicity of irreducibles in Tymoczko’s representation.
Conjecture (Shareshian and MW)

Let $G$ be the incomparability graph of a natural unit interval order. Then

$$\omega X_G(x, t) = \sum_{j=0}^{d(n,r)} \text{ch} H^{2j}(H_G) t^j$$

$H_{G_n,r} = H_{n,r}$. True for $1 \leq r \leq 2$ & $n - 2 \leq r \leq n$.

($r = 2$ follows from results of Procesi and Stanley)

$\Rightarrow$ Schur-unimodality conjecture for $X_G(x, t)$

$\Rightarrow$ $q$-unimodality conjecture for $A_G(q, t)$ (includes $A^{(r)}_n(q, t)$)

$\Rightarrow$ multiplicity of irreducibles in Tymoczko’s representation.

Conjecture

Tymoczko’s representation on $H^{2j}(H_G)$ is a permutation representation in which each point stabilizer is a Young subgroup.

$\Rightarrow$ Stanley-Stembridge e-positivity conjecture for unit interval orders.